Dynkin Diagrams

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1 Introduction

The motivation behind studying Dynkin diagrams is to study the structure of semisimple Lie algebras, that is direct sums of simple Lie algebras. Dynkin diagrams in some sense encode a lot of the structure and properties of semisimple Lie algebras. I will be following the treatments found in Fulton and Harris [1] and Taylor's Lie Group notes [2].

First we start off with a couple of preliminary definitions.

Definition 1. A **simple Lie algebra** $\mathfrak{g}$ is a nonabelian Lie algebra with nonzero proper ideals.

A Lie algebra $\mathfrak{g}$ is called **semisimple** if it is a direct sum of simple Lie algebras.

Definition 2. A **Cartan subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra such that $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$.

Definition 3. We define the **Killing form** B on $\mathfrak{g}$ by

$$B(X, Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)).$$

B defines a symmetric bilinear form on $\mathfrak{g}$.

2 Roots of a Lie algebra

Now given a semisimple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, let us recall what a root system on $\mathfrak{g}$ with relative to $\mathfrak{h}$ is. An element $\alpha \in \mathfrak{h}^*$ is called a **root** if $\alpha \neq 0$ and there exists $X_\alpha \in \mathfrak{g}$ such that

$$[H, X] = \alpha(H)X_\alpha$$

for all $H \in \mathfrak{h}$. Let us call the set of all the roots R. These roots span a real subspace of $\mathfrak{h}^*$ on which the Killing form B is positive definite; call the subspace $\mathbb{E} = \operatorname{span}_{\mathbb{R}} R$. On $\mathbb{E}$, we have a positive definite symmetric bilinear form B, so it makes sense to talk about angles between vectors of $\mathbb{E}$, defined via the Killing form. That is

$$B(u, v) = \cos\theta \frac{||u||}{||v||}.$$

The root system R has the following properties:

(1) R is a finite set spanning $\mathbb{E}$.

(2) If $\alpha \in R$, then $-\alpha \in R$, but no other scalar multiple $c \cdot \alpha \in R$ for $c \neq \pm 1$.

(3) For $\alpha \in R$, the reflection W_α along the hyperplane $\alpha^\perp$ maps R to itself. In fact, W_α is given by

$$W_\alpha(\beta) = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

(4) For $\alpha, \beta \in R$, the number

$$\eta_{\beta,\alpha} = 2\frac{B(\beta, \alpha)}{B(\alpha, \alpha)} \in \mathbb{Z}.$$

For simplicity of notation, we are going abbreviate $B(\cdot\,,\cdot)$ by $(\cdot\,,\cdot)$ and $\eta_{\beta,\alpha}$ by $\eta_{\beta\alpha}$. Anything satisfying properties (1)-(4) is called an **abstract root system**, and in fact these properties will be all we need.

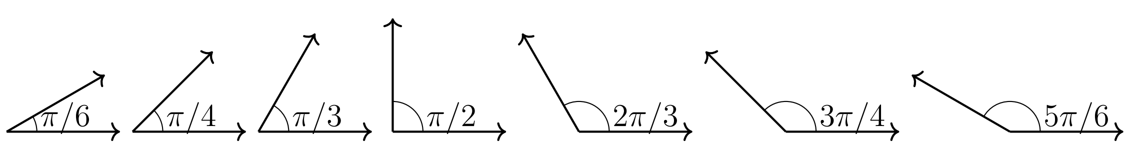
Property (4) implies that

$$\eta_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = 2\cos\theta \frac{||\beta||||\alpha||}{||\alpha||^2} = 2\cos\theta\frac{||\beta||}{||\alpha||} \in \mathbb{Z}.$$

In particular, we have that $\eta_{\alpha\beta}\eta_{\beta\alpha} = 4\cos^2\theta$ is an integer, and in particular, it could only be 0, 1, 2, 3, 4. So even here, we can see that the geometry of root systems is incredibly rigid, since this restricts the allowed angle between roots to a finite set of angles. Now the case $\eta_{\alpha\beta}\eta_{\beta\alpha} = 4$ happens when $\cos^2\theta = 1$, i.e. $\cos\theta = \pm 1$, and thus $\beta = \pm\alpha$, which is a trivial case. So excluding this trivial case, we put the only possible cases in the following table.

$\cos\theta$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$
θ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$\eta_{\beta\alpha}$	3	2	1	0	-1	-2	-3
$\eta_{\alpha\beta}$	1	1	1	0	-1	-1	-1
$\|\|\beta\|\|/\|\|\alpha\|\|$	$\sqrt{3}$	$\sqrt{2}$	1	not defined	1	$\sqrt{2}$	$\sqrt{3}$

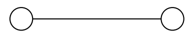
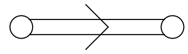
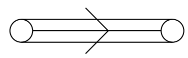
In the table, we picked β and α so that $||\beta|| \geq ||\alpha||$ or $|\eta_{\beta\alpha}| \geq |\eta_{\alpha\beta}|$. Pictorially, we the allowed configurations between each two root must be of one of the following forms.



Now given a root system R and the space it spans $\mathbb{E}$, we can pick a hyperplane not containing any of the roots, say P. P then partitions our root system into two disjoint subsets $R = R_+ \cup R_-$, we call the elements of R_+ the positive roots and elements of R_- the negative roots. We call a root $\alpha \in R_+$ **simple** if it is not the sum of two other positive roots.

3 Defining the Dynkin diagram

We are now in good shape to define the Dynkin diagram of a root system. The Dynkin diagram of a root system consists of nodes ○ which correspond to a simple root, i.e. we draw a "○" for each simple root. We then connect each node with a number of line depending on θ, the angle between each simple root:

No lines		when $\theta = \pi/2$,
One line		when $\theta = 2\pi/3$,
Two lines		when $\theta = 3\pi/4$,
Three lines		when $\theta = 5\pi/6$.

Indeed, if we define $n_{\beta\alpha} = \eta_{\alpha\beta}\eta_{\beta\alpha}$, then the number of lines between two nodes on the Dynkin diagram is exactl $n_{\beta\alpha}$. When we draw the line, we point the arrow from the longer root to the shorter root. There is no arrow when $\theta = 2\pi/3$ because both roots are the same length: $\frac{||\beta||}{||\alpha||} = 1$. One might also notice that we do not have any acute angles on this list; this is because the angle between any two simple root cannot be acute which will follow from axioms of root systems.

Let Σ be the set of positive roots of the root system R. We will now list out some basic facts about root systems.

Proposition 1. *If $\alpha, \beta \in \Sigma$, then neither $\alpha - \beta$ and $\alpha + \beta$ are not roots.*

Proof. If $\alpha - \beta \in R$, then either $\alpha - \beta \in R_+$ or $\alpha - \beta \in R_-$. If $\alpha - \beta \in R_+$, then $\alpha = \beta + (\alpha - \beta)$ is a sum of two positive roots and if $\alpha - \beta \in R_-$, then $\beta - \alpha \in R_+$, so then $\beta = \alpha + (\beta - \alpha)$ is a sum of positive roots. Both cases contradict that α, β are simple roots. $\square$

If α and β are roots with $\beta \neq \pm\alpha$, then we call roots of the form

$$\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha, \ldots, \beta + q\alpha$$

an α-string through β.

Proposition 2. *Given an α string as above, then*

$$p + q \leq 3,$$

and in addition $p - q = \eta_{\beta,\alpha}$.

Proof. Note that we have that

$$W_\alpha(\beta + q\alpha) = \beta - p\alpha,$$

and

$$W_\alpha(\beta + q\alpha) = \left(\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right) - q\alpha,$$

so we must then have that

$$\eta_{\beta,\alpha} + q = p,$$

and thus $p - q = \eta_{\beta,\alpha}$. Next, for $p + q \leq 3$, just take $q = 0$, so then p is an integer no larger than 3. $\square$

Proposition 3. *Suppose α, β are roots with $\beta \neq \pm\alpha$. Then if*

$$(\beta, \alpha) > 0 \implies \alpha - \beta \text{ is a root;}$$
$$(\beta, \alpha) < 0 \implies \alpha + \beta \text{ is a root.}$$

If $(\beta, \alpha) = 0$, then $\alpha - \beta$ and $\alpha + \beta$ are simultaneously roots or nonroots.

Proof. Note that $p - q = \eta_{\beta,\alpha}$, and if $(\beta, \alpha) > 0$, then $\eta_{\beta,\alpha}$ is positive, and consequently, $p - q > 0$, and consequently, $p > 0$, i.e. $q \geq 1$. Similarly, if $(\beta, \alpha) < 0$, then $\eta_{\beta,\alpha}$ is negative, implying that $q > 0$, so $q \geq 1$. $(\beta, \alpha) = 0$, then $p = q$, which gives us what we want. $\square$

Proposition 4. *The angle between two distinct roots cannot be acute, i.e. $(\beta, \alpha) \leq 0$.*

Proof. If $(\beta, \alpha) > 0$, then $\alpha - \beta$ is a root, which contradicts the fact that if α and β are two simple roots then $\alpha - \beta$ cannot be a root. $\square$

Proposition 5. *The simple roots are linearly independent.*

Proof. We will prove this statement using the following statement: If a set of vectors lie on one side of a hyperplane such that all the mutual angles are at least $\pi/2$, then they must all be linearly independent.

Proof of the statement:

Let $v_1, \ldots, v_n$ be vectors on the same side of the hyperplane $ax > 0$, and say that

$$\sum c_i v_i = 0.$$

Then we have that

$$a \sum c_i v_i = c_1(av_1) + \ldots c_n(av_n) = 0.$$

If any c_i is nonzero, then there must be coefficients of both positive and negative signs. Relabel so that $c_1, \ldots, c_k$ are positive and $c_{k+1}, \ldots, c_n$ are negative. Now since $\sum c_i v_i = 0$, we have then that

$$c_1 v_1 + \ldots c_k v_k = -c_{k+1} v_{k+1} - \ldots - c_n v_n.$$

Now, let $v = \sum_{i=1}^{k} c_i v_i$ and $w = -\sum_{i=k+1}^{n} c_i v_i$, but we have that

$$(v, w) = 0,$$

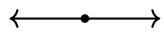
which implies that each $c_i = 0$. This establishes the claim. $\square$

The dimension $n = \dim \mathbb{E} = \dim_{\mathbb{C}} \mathfrak{h}$ is called the **rank** of the root system R. The above claim exactly gives us that $|\Sigma| = n$, i.e. there are exactly n simple roots. Σ forming a basis also tells us that every root $\alpha \in R$ can be written uniquely as a(n integral) linear combination of simple roots. If $\alpha \in R_+$, then we can write α a non-negative integral combination of simple roots. It also follows that no root is a linear combination of simple roots with coefficients of mixed sign.

4 Some examples of root systems

4.1 Rank 1

In the case of rank 1, the only possible root system is the following

.

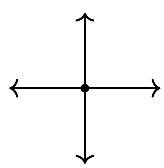
This is the root system of $\mathfrak{sl}_2\,\mathbb{C}$. Label this root system as (A_1). The Dynkin diagram for this root system is simply

$$\circ,$$

i.e. a single node.

4.2 Rank 2

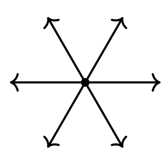
The case that $\theta = \pi/2$, we get the following root system

.

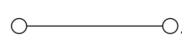
Label this $(A_1 \times A_1)$. This is the root system of $\mathfrak{sl}_2\,\mathbb{C} \times \mathfrak{sl}_2\,\mathbb{C} \simeq \mathfrak{so}_4\,\mathbb{C}$. The Dynkin diagram of this system is

$$\circ \qquad\qquad \circ,$$

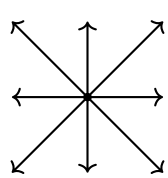
so a disconnected graph with 2 nodes. This example illustrates our next definitions and theorems, which tells us that root systems and Dynkin diagrams encode a lot of the information about semisimple Lie algebras. But before we move on to the next section, let us see some more examples of root systems.



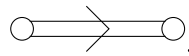
This is called the system (A_2). Its Dynkin diagram is

.

This corresponds to the root system of $\mathfrak{sl}_3\,\mathbb{C}$.



This is the root system (B_2). Its Dynkin diagram is



This is the root system of $\mathfrak{so}_5\,\mathbb{C} \simeq \mathfrak{sp}_4\,\mathbb{C}$.

There is one final rank 2 root system, which we will omit. We move on next section to see why studying root systems give us so much information about the original semistable algebras themselves.

5 The point

Recall from the example of $(A_1 \times A_1)$ that it is the union of two orthogonal sets of roots. This is an illustration of a reducible root system, which define now.

Definition 4. We say that a root system R is **reducible** if it can be decomposed as an orthogonal union of two different root systems. That is, if $R = R_1 \cup R_2$ where for any $\alpha \in R_1$ and $\beta \in R_2$, we have that

$$(\alpha, \beta) = 0.$$

We say that a root system is **irreducible** otherwise.

Now, we can state one of the main important reasons why we care about root systems.

Proposition 6. *A semisimple Lie algebra is simple if and only if its root system is irreducible.*

Proof. A semisimple Lie algebra $\mathfrak{g}$ is simple if and only if the Killing form B is nondegenerate. But the root system of $\mathfrak{g}$ is reducible if and only if B has nontrivial kernel, which establishes our result. □

Corollary 1. *A root system is irreducible if and only if its Dynkin diagram is connected.*

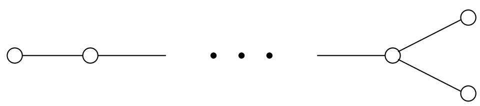
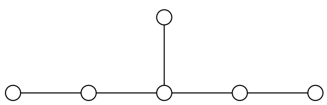
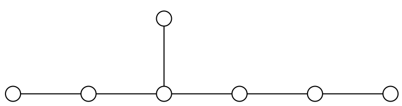
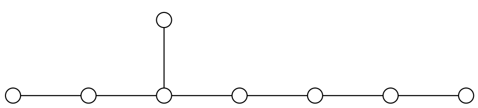
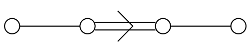
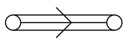
Proof. A Dynkin diagram is connected if and only if there no orthogonal simple roots, which is the previous proposition. □

This means that in some sense, if we can classify Dynkin diagrams, then we can get a lot of information on how to classify semisimple Lie algebras.

6 Classification of Dynkin diagrams

We can classify Dynkin diagrams by classifying all the irreducible Dynkin diagrams, since all Dynkin diagrams will be disjoint unions of these irreducible Dynkin diagrams. We can in fact construct an entire Lie algebra from its Dynkin diagram, but before that, we must first understand the structure of possible Dynkin diagrams.

Theorem 1. *The Dynkin diagrams of irreducible root systems are precisely from the following list (where n in the following is the number of nodes):*

- (A_n): 
- (B_n): 
- (C_n): 
- (D_n): 
- (E_6): 
- (E_7): 
- (E_8): 
- (F_4): 
- (G_2): 

The first four on the list correspond to the following Lie algebras:

$$\begin{array}{ccc} (A_n) & \rightsquigarrow & \mathfrak{sl}_{\mathrm{n+1}}\,\mathbb{C}, \\ (B_n) & \rightsquigarrow & \mathfrak{so}_{\mathrm{2n+1}}\,\mathbb{C}, \\ (C_n) & \rightsquigarrow & \mathfrak{sp}_{\mathrm{2n}}\,\mathbb{C}, \\ (D_n) & \rightsquigarrow & \mathfrak{so}_{\mathrm{2n}}\,\mathbb{C}. \end{array}$$

And then the exceptional cases $(E_6), (E_7), (E_8), (F_4), (G_2)$ all correspond to certain exceptional Lie algebras $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$. We will omit the proof of the theorem as it is too long to present in the scope of this paper.

References

[1] William Fulton and Joe Harris. *Representation Theory: A first course*, volume 129. Springer Science & Business Media, 2013.

[2] Michael Taylor. Lectures on Lie groups. *Lecture Notes, available at http://www. unc. edu/math/Faculty/met/lieg. html*, 4, 2002.