Semigroup theory

Xingyu Cheng

Semigroup theory gives us a way to recast certain PDEs into ODEs. We will begin by setting up some basic theory of (contraction) semigroups, and then highlighting how these techniques can construct a family of solutions to certain second order parabolic PDEs. We will be following the treatment presented in Evans [1].

1 Introduction

We start off with in an abstract setting. Let X be a real Banach space, and we have a linear ODE of the form

$$\begin{cases} u'(t) = Au(t), \\ u(0) = g, \end{cases} \tag{$*$}$$

where $A : D(A) \to X$ is a linear map ($D(A) \subset X$ being the domain of A), $u : \mathbb{R} \to X$ and $g \in X$. Since A is now a linear map of Banach spaces, we could have A be some sort of differential operator. In such a case, we can recast many PDEs into the form of $(*)$. Our key problems are then to see what conditions we need so that we have existence and uniqueness of solutions to $(*)$ and when and how can we cast a PDE into the form $(*)$.

Now classically speaking, if $X = \mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^n$ linear, $u : \mathbb{R} \to \mathbb{R}^n$, and $g = u_0 \in \mathbb{R}^n$ an initial value vector, then we have that solutions to $(*)$ look like $u(t) = e^{At}u_0$. Morally, we also want solutions to $(*)$ to be of a similar form in the general setting that we have now. So for now, let us informally assume that $(*)$ has a unique solution $u(t) = S(t)g$ for each initial point $g \in X$ and $t \geq 0$. For each t, we can view $S(t)$ as a map $X \to X$. Again, morally speaking, we want our operators $S(t)$ to satisfy properties similar to the exponentials 'e^{At}.' That is, we want the family $\{S(t)\}_{t\geq 0}$ to satisfy:

(1) Each $S(t) : X \to X$ is linear.

(2) $S(0)$ is the identity operator, i.e. $S(0)g = g$ for all $g \in X$.

(3) $S(s + t)g = S(s)S(t)g = S(t)S(s)g$ for $s, t \geq 0, g \in X$.

(4) The map $t \mapsto S(t)$ is continuous for each t.

Notice now the operators $\{S(t)\}_{t\geq 0}$ now inherit algebraic properties similar to the non-negative reals $[0, \infty)$ by properties (2) and (3).

Definition 1. a) A family of operators $\{S(t)\}_{t\geq 0}$ is a **semigroup** if it satisfies conditions (1)–(4) above.

b) If in addition each $S(t)$ satisfies $||S(t)|| \leq 1$ where $|| \cdot ||$ is the operator norm, then we say that $\{S(t)\}_t$ is a **contraction semigroup.**

The name semigroup comes from the fact that the family $\{S(t)\}_{t\geq 0}$ forms a semigroup in the algebraic sense: associative multiplication, but no inverse. In fact, property (2) actually ensures the we have a monoid. It also turns out that contraction semigroups are the actual objects we want to think about.

2 First properties of contraction semigroups

From here on, assume that $S = \{S(t)\}_{t\geq 0}$ is a contraction semigroup on X. Note that since each $||S(t)|| \leq 1$, each $S(t)$ is a bounded, i.e. continuous operator on X and hence $S(t)$ commutes with limits for all $t \geq 0$.

Definition 2. Define

$$D(A) = \left\{ g \in X \,\middle|\, \lim_{t\to 0^+} \frac{S(t)g - g}{t} \text{ exists in } X \right\}$$

and define $A : D(A) \to X$ by

$$Ag = \lim_{t\to 0^+} \frac{S(t)g - g}{t}.$$

We call A the (infinitesimal) generator of the semigroup S.

Theorem 1 (Differential properties of semigroups)**.** *Assume $g \in D(A)$, then*

a) $S(t)g \in D(A)$ for all $t \geq 0$,

b) $AS(t)g = S(t)Ag$ for all $t \geq 0$,

c) $t \mapsto S(t)g$ is differentiable for all $t > 0$, and

d) $\frac{d}{dt}S(t)g = AS(t)g$.

Proof. For $S(t)g \in D(A)$, simply look at the limit

$$\lim_{s\to 0^+} \frac{S(s)S(t)g - S(t)g}{s}. \tag{1}$$

Now since $S(s)S(t) = S(t)S(s)$, we have that (1) becomes

$$\begin{aligned}\lim_{s\to 0^+} \frac{S(t)S(s)g - S(t)g}{s} &= \lim_{s\to 0^+} S(t) \left(\frac{S(s)g - g}{s} \right) \\ &= S(t) \lim_{s\to 0^+} \frac{S(s)g - g}{s} = S(t)Ag.\end{aligned}$$

Now $S(t)Ag \in X$ exists since $g \in D(A)$ and $D(S(t)) = X$, so indeed $S(t)g \in D(A)$. Notice also that (1) is exactly the formula for $AS(t)g$, so this also gives us that $AS(t) = S(t)A$.

Now we need to see that $t \mapsto S(t)g$ is differentiable with the formula that we want. From the argument above, we have $\lim_{h\to 0^+} \frac{S(t+h)g - S(t)g}{h} = S(t)Ag$, so we need only see what happens when

$h \to 0^-$ of the above. But we can also recast limit as $\lim_{h\to 0^+} \frac{S(t)g - S(t-h)g}{h}$, and we want to see that the above limit goes to $S(t)Ag$. But this is the same as asking that

$$\lim_{h\to 0^+} \left(\frac{S(t)g - S(t-h)g}{h} - S(t)Ag \right) = 0. \tag{2}$$

Now notice that $S(t) = S(t-h+h) = S(t-h)S(h)$, so we have that

$$\frac{S(t)g - S(t-h)g}{h} = S(t-h)\frac{S(h)g - g}{h},$$

and that $S(t) = S(t) - S(t-h) + S(t-h)$, which gives us that $S(t)Ag = S(t)Ag - S(t-h)Ag + S(t-h)Ag$. Putting everything together, we get that (2) becomes

$$\begin{aligned}
&\lim_{h\to 0^+} \left[\left(S(t-h)\frac{S(h)g - g}{h} - S(t-h)Ag \right) + (S(t-h) - S(t))Ag \right] \\
&\quad = \lim_{h\to 0^+} \left[S(t-h) \left(\frac{S(h)g - g}{h} - Ag \right) + (S(t-h) - S(t))Ag \right]
\end{aligned}$$

Now since $t \mapsto S(t)$ is continuous, we have that $S(t-h) \to S(t)$ as $h \to 0$ (for $t > 0$), and similarly, we know already that $(S(h)g - g)/h \to Ag$ as $h \to 0^+$, which means that that above limit goes to 0 as $h \to 0^+$. Consequently, we have that

$$\lim_{h\to 0^+} \frac{S(t)g - S(t-h)g}{h} - S(t)Ag = 0.$$

This tells us then that

$$\lim_{h\to 0} \frac{S(t+h)g - S(t)g}{h} = S(t)Ag = AS(t)g,$$

which shows us both that the derivative exists and has the formula we want. □

Remark. The map $t \mapsto AS(t)g$ is continuous, which implies that the map $t \mapsto S(t)g$ is a C^1 map for $t \in (0, \infty)$.

Theorem 2 (Properties of generators)**.** *a) The domain $D(A)$ is dense in X.*

b) A is a closed operator.

(A closed means that if (g_k) is a sequence in $D(A)$ and $g_k \to g$, $Ag_k \to v$, then $g \in D(A)$ and $v = Ag$.)

Proof. For $g \in X$, define $g^t = \int_0^t S(s)g\,ds$. Since $s \mapsto S(s)g$ is continuous, the limit

$$\lim_{t\to 0^+} \frac{g^t}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t S(s)g\,ds = S(0)g = g.$$

Now the claim is that $g^t \in D(A)$ for $t > 0$. If $r > 0$, we have that

$$\begin{aligned}
\frac{S(r)g - g}{r} &= \frac{1}{r}\left[S(r)\left(\int_0^t S(s)g\,ds \right) - \int_0^t S(s)g\,ds \right] \\
&= \frac{1}{r}\left[\left(\int_0^t S(r+s)g\,ds \right) - \int_0^t S(s)g\,ds \right].
\end{aligned}$$

Now a change of variables on $\int_0^t S(r+s)g\,ds$ gives $\int_r^{t+r} S(s)g\,ds = \int_0^{t+r} S(s)g\,ds - \int_0^r S(s)g\,ds$, so the above equation becomes

$$\frac{1}{r}\int_t^{t+r} S(s)g\,ds - \frac{1}{r}\int_0^r S(s)g\,ds.$$

Letting $r \to 0$, the above becomes $S(t)g - g$. Thus $g^t \in D(A)$ and $Ag^t = S(t)g - g$.

Next, we want that A is a closed operator. Let (g_k) be a sequence in $D(A)$ such that $g_k \to g$ and $Ag_k \to v$ in X. We need to see that $g \in D(A)$ and $Ag = v$. Now, since $\frac{d}{dt}S(t)g = S(t)Ag$, we have that

$$\int_0^t S(s)Ag_k\,ds = S(t)g_k - S(0)g_k = S(t)g_k - g_k.$$

Now letting $k \to \infty$, we get

$$S(t)g - g = \int_0^t S(s)v\,ds.$$

So then dividing by t and taking the limit, we see that

$$\lim_{t\to 0^+}\frac{S(t)g-g}{t} = \lim_{t\to 0^+}\frac{1}{t}\int_0^t S(s)v\,ds = v.$$

Now by definition of $g \in D(A)$, we get that $Ag = v$ as desired. □

3 Resolvents

Let $A : D(A) \to X$ be a closed linear operator on X.

Definition 3. a) A real number $\lambda \in \rho(A)$, the **resolvent set** of A if the operator

$$\lambda I - A : D(A) \to X$$

is bijective.

b) If $\lambda \in \rho(A)$, the **resolvent operator** $R_\lambda : X \to X$ is defined by

$$R_\lambda g = (\lambda I - A)^{-1} g.$$

The Closed Graph Theorem guarantees that $R_\lambda : X \to D(A) \subset X$ is a bounded linear operator. Furthermore, $R_\lambda A g = A R_\lambda g$ for all $g \in D(A)$.

Theorem 3 (Properties of resolvent operators)**.** *a) If $\lambda, \mu \in \rho(A)$, we have then*

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu \quad \text{and} \quad R_\lambda R_\mu = R_\mu R_\lambda.$$

b) If $\lambda > 0$, then $\lambda \in \rho(A)$,

$$R_\lambda g = \int_0^\infty e^{-\lambda t} S(t)g\,dt$$

and so $||R_\lambda|| \leq \frac{1}{\lambda}$.

Proof. a)

$$\begin{aligned}
R_\lambda - R_\mu &= (\lambda I - A)^{-1} - (\mu I - A)^{-1} \\
&= (\lambda I - A)^{-1}(\mu I - A)(\mu I - A)^{-1} - (\lambda I - A)^{-1}(\lambda I - A)(\mu I - A)^{-1} \\
&= (\lambda I - A)^{-1}((\lambda I - A) - (\mu I - A))(\mu I - A)^{-1} \\
&= R_\lambda((\lambda - \mu)I)R_\mu = (\lambda - \mu)R_\lambda R_\mu.
\end{aligned}$$

To see that $R_\lambda R_\mu = R_\mu R_\lambda$, notice that we can multiply 1 in a different order (from the outside instead of the inside) to get that $R_\lambda - R_\mu = (\lambda - \mu)R_\mu R_\lambda$, so we have that

$$(\lambda - \mu)R_\lambda R_\mu = (\lambda - \mu)R_\mu R_\lambda.$$

If $\lambda = \mu$, then the commutation is immediate, but if $\lambda \neq \mu$, then we can divide to get what we want.

b) We have that $\lambda > 0$ and $||S(t)|| < 1$, so we have that the integral

$$\int_0^\infty e^{-\lambda t} S(t) g \, dt$$

is defined. Let $\tilde{R}_\lambda g$ be the integral. Then for $h > 0$, $g \in X$, we have that

$$\frac{S(h)\tilde{R}_\lambda g - \tilde{R}_\lambda g}{h} = \frac{1}{h}\left(\int_0^\infty e^{-\lambda t}[S(t+h)g - S(t)g] \, dt\right).$$

Now looking at

$$\int_0^\infty e^{-\lambda t} S(t+h) g \, dt,$$

a change of variables gives us that this is equal to

$$\int_h^\infty e^{-\lambda(t-h)} S(t) g \, dt = \int_0^\infty e^{-\lambda(t-h)} S(t) g \, dt - \int_0^h e^{-\lambda(t-h)} S(t) g \, dt.$$

Substituting in to the equation, we get

$$\begin{aligned}
\frac{1}{h}\int_0^\infty (e^{-\lambda(t-h)} - e^{-\lambda t}) S(t) g \, dt &- \frac{1}{h}\int_0^h e^{-\lambda(t-h)} S(t) g \, dt \\
&= \frac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda t} S(t) g \, dt - e^{\lambda h}\frac{1}{h}\int_0^h e^{-\lambda t} S(t) g \, dt.
\end{aligned}$$

Taking the limit as $h \to 0^+$ we see that

$$A\tilde{R}_\lambda g = \lim_{h \to 0^+} \frac{S(h)\tilde{R}_\lambda g - \tilde{R}_\lambda g}{h} = \lambda \tilde{R}_\lambda g - g,$$

where $\frac{e^{\lambda h} - 1}{h} \to \lambda$ by l'Hôpital's Rule. So we get that $g = (\lambda I - A)\tilde{R}_\lambda g$.

Now, if $g \in D(A)$, we have that

$$A\tilde{R}_\lambda g = A\int_0^\infty e^{-\lambda t} S(t) g \, dt.$$

The claim now is that

$$A \int_0^\infty e^{-\lambda t} S(t) g \, dt = \int_0^\infty e^{-\lambda t} A S(t) g \, dt.$$

First consider

$$A \int_0^k e^{-\lambda t} S(t) g \, dt.$$

We want to see that $A \int_0^k e^{-\lambda t} S(t) g \, dt = \int_0^k e^{-\lambda t} A S(t) g \, dt$. Approximate $\int_0^k e^{-\lambda t} S(t) g \, dt$ by a Riemann sum

$$\sum_{i=1}^N e^{-\lambda \tau_i} (t_i - t_{i-1}) S(\tau_i) g,$$

where $P_N = (0 = t_0, t_1, \ldots, t_N = k)$ is a partition of $[0, k]$, and each $\tau_i \in [t_i, t_{i-1}]$. Now, we have that since each $S(\tau_i) g \in D(A)$ and $D(A)$ is a vector subspace of X, the Riemann sum is in $D(A)$ as well, so we have that

$$A \left(\sum_{i=1}^N e^{-\lambda \tau_i} (t_i - t_{i-1}) S(\tau_i) g \right) = \sum_{i=1}^N e^{-\lambda \tau_i} (t_i - t_{i-1}) A S(\tau_i) g$$

makes sense and is justified. Now, by taking the limit of finer and finer partitions (so say we have a sequence of partitions (P_N) and we take the limit as $N \to \infty$), we have that

$$\sum_{i=1}^N e^{-\lambda \tau_i} (t_i - t_{i-1}) S(\tau_i) g \to \int_0^k e^{-\lambda t} S(t) g \, dt,$$

and

$$\sum_{i=1}^N e^{-\lambda \tau_i} (t_i - t_{i-1}) A S(\tau_i) g \to \int_0^k e^{-\lambda t} A S(t) g \, dt.$$

A is a closed operator, so indeed,

$$A \int_0^k e^{-\lambda t} S(t) g \, dt = \int_0^k e^{-\lambda t} A S(t) g \, dt.$$

Now, letting $k \to \infty$, we get then that by again using the fact that A is a closed operator

$$A \int_0^\infty e^{-\lambda t} S(t) g \, dt = \int_0^\infty e^{-\lambda t} A S(t) g \, dt.$$

So we have then that

$$A \tilde{R}_\lambda g = \int_0^\infty e^{-\lambda t} A S(t) g \, dt = \int_0^\infty e^{-\lambda t} S(t) A g \, dt = \tilde{R}_\lambda A g.$$

Thus we have that $\tilde{R}_\lambda (\lambda I - A) g = g$ for $g \in D(A)$. So now since we have that

$$\tilde{R}_\lambda (\lambda I - A) g = g \quad \text{and} \quad (\lambda I - A) \tilde{R}_\lambda g = g,$$

we get then that $(\lambda I - A)$ is a bijection with inverse $\tilde{R}_\lambda = R_\lambda$ and therefore $\lambda \in \rho(A)$. $\square$

4 Characterization of generators of contraction semigroups

The following theorem characterizes generators of contraction semigroups.

Theorem 4 (Hille-Yosida)**.** *Let A be a closed, densely defined linear operator on X. A is the generator of a contraction semigroup $\{S(t)\}_{t\geq 0}$ if and only if*

$$(0,\infty)\subset\rho(A)\quad \textit{and}\quad ||R_\lambda||\leq\frac{1}{\lambda}\textit{ for } \lambda>0.$$

Proof. The forward direction (i.e. assuming A generates a semigroup) follows from the previous theorem.

So now suppose A is closed, densely defined linear operator on X and $(0,\infty)\subset\rho(A)$ and $||R_\lambda||\leq 1/\lambda$ for $\lambda>0$. We want to build a contraction semigroup with A as its generator. Define $A_\lambda=-\lambda I+\lambda^2R_\lambda=\lambda AR_\lambda$ (the last equality coming from $(\lambda^2I-\lambda A)R_\lambda=\lambda I$). This operator A_λ will be our approximation for A.

For A_λ to be an approximation, we should have that $A_\lambda g\to Ag$ as $\lambda\to\infty$ for $g\in D(A)$. Now since $\lambda AR_\lambda g-g=AR_\lambda g=R_\lambda Ag$, and $||\lambda R_\lambda g-g||\leq||R_\lambda||||Ag||\leq\frac{1}{\lambda}||Ag||$, we have then letting $\lambda\to\infty$, $\lambda AR_\lambda g\to g$ for $g\in D(A)$. Now, since $||\lambda R_\lambda||\leq 1$ (so R_λ is continuous) and $D(A)$ is dense, we deduce then that $\lambda R_\lambda g\to g$ as $\lambda\to\infty$ for all $g\in X$ (by first approximating g and then letting $\lambda\to\infty$). Now if $g\in D(A)$, then we have that $A_\lambda g=\lambda AR_\lambda g=\lambda R_\lambda Ag\to Ag$ as $\lambda\to\infty$, which is what we want.

Now that we have an approximation for A, we want to somehow go from A to get our semigroup element $S(t)$. To do that, we are going to go first from A_λ to an operator $S_\lambda(t)$. Define $S_\lambda(t)=e^{tA_\lambda}=e^{t(-\lambda I+\lambda^2R_\lambda)}$. We can then rewrite

$$e^{t(-\lambda I+\lambda^2R_\lambda)}=e^{-\lambda t}e^{\lambda^2tR_\lambda}=e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda^2t)^k}{k!}R_\lambda^k.$$

Now, since $||R_\lambda||\leq\frac{1}{\lambda}$, we have then that

$$||S_\lambda(t)||\leq e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda^2t)^k}{k!}\frac{1}{\lambda^k}=\sum_{k=0}^{\infty}\frac{\lambda^kt^k}{k!}=e^{-\lambda t}e^{\lambda t}=1.$$

Thus, we get a contraction semigroup $\{S_\lambda(t)\}_{t\geq 0}$ from A_λ. The semigroup is generated by A_λ with $D(A_\lambda)=X$ (since $\lim_{t\to 0^+}\frac{S_\lambda(t)g-g}{t}=\lim_{t\to 0^+}\frac{e^{tA_\lambda}g-g}{t}=A_\lambda g$ for all $g\in X$).

Let $\lambda,\mu>0$ (and so $\lambda,\mu\in\rho(A)$), and so $R_\lambda R_\mu=R_\mu R_\lambda$. Because of this commutation relationship, and $A_\lambda=\lambda AR_\lambda$, we have then that $A_\lambda A_\mu=A_\mu A_\lambda$ (since everything commutes in the expression). Since A_λ and A_μ commute, we have then that $A_\mu S_\lambda(t)=S_\lambda(t)A_\mu$ for each $t>0$. Now, we want to compute $S_\lambda(t)g-S_\mu(t)g$. Notice now that

$$\int_0^t\frac{d}{ds}(S_\mu(t-s)S_\lambda(s)g)\,ds=S_\mu(t-t)S_\lambda(t)g-S_\mu(t-0)S_\lambda(0)g=S_\lambda(t)g-S_\mu(t)g.$$

Now,

$$\frac{d}{ds}(S_\mu(t-s)S_\lambda(s)g)=-S'_\mu(t-s)S_\lambda(s)g+S_\mu(t-s)S'_\lambda(s)g,$$

and since $S'_\lambda(t) = S_\lambda(t)A_\lambda g$ and everything commute, we have that

$$-S'_\mu(t-s)S_\lambda(s)g + S_\mu(t-s)S'_\lambda(s)g = S_\mu(t-s)S_\lambda(s)A_\lambda g - S_\mu(t-s)S_\lambda(s)A_\mu g,$$

and thus

$$S_\lambda(t)g - S_\mu(t)g = \int_0^t S_\mu(t-s)S_\lambda(s)(A_\lambda g - A_\mu g)\, ds.$$

Taking norms of both sides, we see then that

$$||S_\lambda(t)g - S_\mu(t)g|| \leq \int_0^t ||S_\mu(t-s)||||S_\lambda(s)||||A_\lambda g - A_\mu g||\, ds \leq t||A_\lambda g - A_\mu g||.$$

Now because $A_\lambda g \to Ag$ for each $g \in D(A)$, we have then that $||A_\lambda g - A_\mu g|| \to 0$ as $\lambda, \mu \to \infty$. This means then that for each $t \geq 0$, the limit $\lim_{\lambda\to\infty} S_\lambda(t)g$ exists for $g \in D(A)$. Call this limit $S(t)g$. Since $||S_\lambda(t)|| \leq 1$, the limit exists for all $g \in X$ and uniformly for t on compact subsets of $[0, \infty)$. $\{S_\lambda(t)\}_{t\geq 0}$ is also a contraction semigroup on X since we can take the super over $||g|| \leq 1$ on

$$||S(t)g||_X = \lim_{\lambda\to\infty} ||S_\lambda(t)g||_X$$

and see that indeed $||S(t)|| \leq 1$.

Finally, we need to now show that A is the generator of $\{S(t)\}_{t\geq 0}$. Let B be the generator of $\{S(t)\}_{t\geq 0}$. We want to then see that $A = B$. First, note that

$$||S_\lambda(s)A_\lambda g - S(s)A_\lambda g|| \leq ||S_\lambda(s)||||A_\lambda g - Ag|| + ||(S_\lambda(s) - S(s))Ag||,$$

and since $||A_\lambda g - Ag|| \to 0$ as $\lambda \to \infty$ for $g \in D(A)$ and $||S_\lambda(s) - S(s)|| \to 0$ as $\lambda \to \infty$, we have then that the right hand side of the above inequality goes to 0 as $\lambda \to \infty$ for $g \in D(A)$, and thus $S_\lambda(s)A_\lambda g \to S(s)Ag$. Next, note that we have

$$S_\lambda(t)g - g = \int_0^t \frac{d}{ds}(S_\lambda(s)g)\, ds = \int_0^t S_\lambda(s)A_\lambda g\, ds.$$

Taking the limit as $\lambda \to \infty$, we get then that

$$S(t)g - g = \int_0^t S(s)Ag\, ds$$

for $g \in D(A)$. This means that the limit

$$Bg = \lim_{t\to 0^+} \frac{S(t)g - g}{t} = Ag$$

for $g \in D(A)$ and so $D(A) \subset D(B)$. Now if $\lambda > 0$, then $\lambda \in \rho(A) \cap \rho(B)$. $(\lambda I - B)(D(A)) = (\lambda I - A)(D(A)) = X$. So we have that $(\lambda I - B)\mid_{D(A)}$ is bijective and from here we deduce that $D(A) = D(B)$. Therefore $A = B$, i.e. A is the generator of $\{S(t)\}_{t\geq 0}$. $\square$

5 Application to second order parabolic PDEs

Now that we have some theory about semigroups, we can now move on to apply them to study second order parabolic PDEs. Consider the following boundary value problem

$$\begin{cases} u_t + Lu = 0 & \text{in } U \times [0,T], \\ u = 0 & \text{on } \partial U \times [0,T], \\ u = g & \text{on } U \times \{t=0\}, \end{cases} \tag{$**$}$$

where U is a bounded open set with smooth boundary. Assume that L has the divergence form

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u,$$

and that it satisfies the uniform ellipticity condition and has smooth coefficients that do not depend on t. We are going to use semigroup methods to study such an equation.

To use semigroup methods, first let $X = L^2(U)$, and set $D(A) = H_0^1(U) \cap H^2(U)$ and define $Au = -Lu$ for $u \in D(A)$. Now A is an unbounded linear operator on X. We have an energy estimate

$$\beta ||u||^2_{H_0^1(U)} \leq B(u,u) + \gamma ||u||^2_{L^2(U)}$$

for constants $\beta > 0$, $\gamma \geq 0$ and B the bilinear form associated to L.

Definition 4. Let $\gamma \in \mathbb{R}$. A semigroup $\{S(t)\}_{t\geq 0}$ is called γ-contractive if $||S(t)|| \leq e^{\gamma t}$ for $t \geq 0$.

Using similar methods as the Hille-Yosida Theorem, it can be seen that a closed, densely define operator A generated a γ-contractive semigroup if and only if

$$(\gamma, \infty) \subset \rho(A) \quad \text{and} \quad ||R_\lambda|| \leq \frac{1}{\lambda - \gamma} \text{ for all } \lambda > \gamma.$$

Theorem 5 (Second-order parabolicc PDE as semigroups)**.** *The operator A generates a γ-contraction semigroup on $L^2(U)$.*

Proof. We need to see that A satisfies the conditions above. The $D(A)$ given is clearly dense in $L^2(U)$, so we need now that A is closed. Let (u_k) be a sequence in $D(A)$ with $u_k \to u$ and $Au_k \to f$ in $L^2(U)$. Now we have the following H^2 estimate on the u_k's:

$$||u_m - u_n||_{H^2} \leq C(||Au_m - Au_n||_{L^2} + ||u_m - u_n||_{L^2})$$

(where all the spaces are restricted to U). This holds for all m and n. Now, since $u_k \to u$ and $Au_k \to f$ in L^2, this means then that the sequence (u_k) is Cauchy in H^2 and so $u_k \to u$ in H^2. Therefore $u \in D(A)$. Furthermore, since $u_k \to u$, we have that $Au_k \to Au$ in L^2 and so $f = Au$. So indeed A is a closed operator.

We next need to check the resolvent conditions. Now we know that for each $\lambda \geq \gamma$, the boundary value problem

$$\begin{cases} Lu + \lambda u = f & \text{in } U, \\ u = 0 & \text{in } \partial U \end{cases}$$

has a unique weak solution $u \in H_0^1(U)$ for each $f \in L^2(U)$. By elliptic regularity, we have that u must also be in H^2, and so $u \in H^2 \cap H_0^1 = D(A)$. We can then rewrite the above equation to be

$$\lambda u - Au = f$$

using the fact that we defined $Au = -Lu$. This implies that $\lambda I - A : D(A) \to X$ is a bijection for $\lambda \geq \gamma$, and so $(\gamma, \infty) \subset \rho(A)$ as wanted.

Now we want the bound on $||R_\lambda||$. To do that, consider the weak problem

$$B(u, v) = \lambda(u, v) = (f, v)$$

for each $v \in H_0^1(U)$, where $(\cdot, \cdot)$ is the $L^2(U)$ pairing. Set $u = v$ and use the energy estimate to get that for $\lambda > \gamma$

$$(\lambda - \gamma)||u||_{L^2(U)}^2 \leq ||f||_{L^2(U)}||u||_{L^2(U)}.$$

Since $u = R_\lambda f$, we have the estimate

$$||R_\lambda f||_{L^2} \leq \frac{1}{\lambda - \gamma}||f||_{L^2}$$

Taking the sup over $||f||_{L^2} \leq 1$, we get then that $||R_\lambda|| \leq \frac{1}{\lambda - \gamma}$ as desired. □

As we can see here, semigroup theory tells us how find a family of solutions to such parabolic PDEs. Furthermore, from the proof of the Hille-Yoside Theorem, we actually have a very explicit way of constructing solutions to such PDEs.

6 My additions

In the the Introduction, I added clarification of the abstract setting with the example of ODEs in $\mathbb{R}^n$, which highlights how semigroup theory is really a way to make sense of 'e^{At}' but in the more abstract setting. Right after the definition of semigroup, I added clarified the reason for the name semigroup. I also added additional justification on why $S(t)$ commute with all limits as the first sentence in Section 2. In the proof of Theorem 2, I added additional clarification about the change of variables employed in the proof of part a). In the proof of Theorem 3, I supplied the proof of part a). Furthermore, in the proof of part b), I supplied the justification of $A \int_0^\infty e^{-\lambda t} S(t)g\, dt = \int_0^\infty e^{-\lambda t} AS(t)g\, dt$ via Riemann sums.

References

[1] Lawrence C Evans. *Partial Differential Equations*, volume 19. American Mathematical Soc., 2010.