Module of differentials

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1 Introduction

Module of differentials are a way of bringing calculus techniques into the world of algebraic geometry. In manifold theory, we often 'linearize' manifolds by considering vector bundles on the manifold. In particular, the tangent and cotangent bundles of a manifold are of particular interest. In algebraic geometry, we do not have a baked-in notion of differentiability and so it takes a bit more work to get these calculus techniques to work. Here, I will lay out the basic notions of modules of differentials along with some first properties. I will be following the treatments found in Eisenbud [1] and I will sometimes refer to Hartshorne [2] for certain insights from geometry.

2 Definitions and basic notions

Definition 1. Let S be a ring, M an S-module. A map (of abelian groups)

$$d : S \to M$$

is called a **derivation** if it satisfies the *Leibniz rule*

$$d(fg) = d(f)g + fd(g) \quad \text{for } f, g \in S.$$

If S is an R-algebra, then d is called R-linear if it is in addition a map of R-modules. We will note the set of all R-linear derivations $S \to M$ by $\mathrm{Der}_R(S, M)$.

The set $\mathrm{Der}_R(S, M)$ carries a natural S-module structure by

$$s \cdot d : f \mapsto s \cdot d(f) \quad \text{for } s \in S, d \in \mathrm{Der}_R(S, M), f \in S.$$

The following are some familiar examples of derivations that we have seen before:

$d : C^1(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$,

any vector field X on $\mathbb{R}^n$,

any vector field X on a smooth manifold M,

$d : C^\infty(M) \to \Omega^1(M)$ for some smooth manifold M.

In the algebra setting, we could take $S = M = k[x, y]$ for example, and we have a derivation given naturally by $\frac{\partial}{\partial x} : k[x, y] \to k[x, y]$. This derivation is $k[y]$=linear, so $\partial/\partial x \in \mathrm{Der}_{k[y]}(k[x, y], k[x, y])$. We further have that $\mathrm{Der}_{k[y]}(k[x, y], k[x, y])$ is a free $k[y]$-module of rank 1, generated by $\partial/\partial x$.

Now in the geometric case, given an affine variety $X \subset \mathbb{A}^n$, we can take its coordinate ring $S = k[X] = k[x_1, \dots, x_r]/I(X)$. It turns out then that $\mathrm{Der}_k(S, S)$ will be the set of algebraic tangent vector fields on X, i.e. sections of the 'tangent bundle' TX.

For any derivation d, we have that $d(1) = d(1 \cdot 1) = d(1)1 + 1d(1) = 2d(1)$, and hence $d(1) = 0$. Further, d is R-linear if and only if $d(a) = 0$ for all $a \in R$.

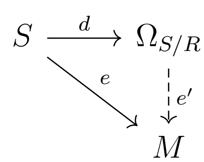
Definition 2. If S is an R-algebra, then the **module of Kähler differentials** of S over R, written $\Omega_{S/R}$ is the S-module generated by the set $\{d(f) \mid f \in S\}$ subject to the relations

$$\begin{aligned} d(fg) &= d(f)g + fd(g) \quad \text{(Leibniz rule)}, \\ d(af + bg) &= ad(f) + bd(g) \quad (R\text{-linearity}) \end{aligned}$$

for $a, b \in R$, $f, g \in S$. We will often abbreviate df instead of $d(f)$. The R-linear derivation $d : S \to \Omega_{S/R}$ is called the **universal R-linear derivation**.

Remark. This should really remind you of differential forms.

The universal derivation and $\Omega_{S/R}$ satisfy the following universal property: Given any S-module M and R-linear derivation $e : S \to M$, there is a unique S-linear morphism $e' : \Omega_{S/R} \to M$ such that the following diagram commutes.



So this means that we have an isormorphism $\mathrm{Der}_R(S, M) \simeq \mathrm{Hom}_S(\Omega_{S/R}, M)$, and so really studying derivations from S to M is the same as studying the module of differentials.

Viewing the module of differentials in the geometry world, we have that R would correspond to some base scheme X an S would correspond to a scheme Y over X, and $R \to S$ becomes $Y \to X$. Now instead of a module of differentials, we now have a sheaf $\Omega_{Y/X}$ of relative differentials on Y. Now this is analogous to the case in smooth manifolds where we have the sheaf Ω^1 of differential 1-forms on M.

If S is generated as an R-algebra by elements $f_1, \dots, f_s$, then $\Omega_{S/R}$ would be generated by $df_1, \dots, df_s$. If $f = F(f_1, \dots, f_s)$ is an element of S, where $F \in R[x_1, \dots, x_s]$, then by repeatedly using the product rule, we find that df is a linear combination of the df_i's. This tells us that $\Omega_{S/R}$ is a finitely generated S module when S is a finitely generated R-algebra, despite the fact that we defined $\Omega_{S/R}$ as a quotient of a module with very many generators. The above argument gives us our first example, also.

Example 1. If $S = R[x_1, \dots, x_r]$ is the polynomial ring over R with r variables, then $\Omega_{S/R} = \bigoplus_{i=1}^r S dx_i$, the free module generated by the dx_i's.

Proof. Since S is generated as an R-algebra by the x_i's, $\Omega_{S/R}$ is generated as an S-module by the dx_i and there is a surjection $S^r \to \Omega_{S/R}$ taking the i-th basis of S^r to dx_i (by the above argument).

Now for the other direction, note that $\partial/\partial x_i$ is an R-linear derivation $S \to S$ and thus by universal property we have an S-module map $\partial_i : \Omega_{S/R} \to S$ mapping dx_i to 1 and the other dx_j to 0. Putting all these maps together, we get the inverse map

$$\Omega_{S/R} \xrightarrow{\begin{pmatrix} \partial_1 \\ \vdots \\ \partial_r \end{pmatrix}} S^r.$$

□

2.1 Functorial properties of differentials

Going from an algebra S over R to the module of differentials $\Omega_{S/R}$ is actually functorial in the following sense. We start off with the category of objects that are the following diagram $\begin{array}{c} S \\ \uparrow \\ R \end{array}$, i.e. algebras over a ring, and then the morphisms in this category are commutative diagrams of the following form $\begin{array}{ccc} S & \xrightarrow{\varphi} & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$. The functor then sends objects to $\begin{array}{c} \Omega_{S/R} \\ d\uparrow \\ S \end{array}$, and the commutative diagrams to $\begin{array}{ccc} \Omega_{S/R} & \longrightarrow & \Omega_{S'/R'} \\ d\uparrow & & d'\uparrow \\ S & \xrightarrow{\varphi} & S' \end{array}$, where the lower arrow φ is the same as the φ above, and the S-linear morphism $\Omega_{S/R} \to \Omega_{S'/R'}$ is the one induced by the derivation $d'\varphi$.

Often times in the wild, we will often just take $R \to R'$ to be the identity (i.e. we can just talk about algebras over k), and we can just think about morphisms of R-algebras $S \to S'$. In this case, we just talk about taking S to $\Omega_{S/R}$. We can also replace the data of the S-linear morphism $\Omega_{S/R} \to \Omega_{S'/R}$ with the equivalent date of a S'-linear morphism $S' \otimes \Omega_{S/R} \to \Omega_{S'/R}$. Also we might also suppress mentioning the universal derivation d and just just talk about $\Omega_{S/R}$ as a functor itself. This functor is often called the **relative cotangent functor**. This functor is right exact.

Proposition 1 (Relative Cotangent Sequence). *If $R \to S \to T$ are maps of rings, then there is a right-exact sequence of T-modules*

$$T \otimes_S \Omega_{S/R} \to \Omega_{T/R} \to \Omega_{T/S} \to 0$$

where $\Omega_{T/R} \to \Omega_{T/S}$ is given by $df \mapsto df$, and the map $T \otimes_S \Omega_{S/R} \to \Omega_{T/R}$ is given by $a \otimes df \mapsto adf$.

Proof. Notice that $\Omega_{T/S}$ and $\Omega_{T/R}$ have the same generators except that $\Omega_{T/S}$ have more relations given by $df = 0$ for $f \in S$. But now look at the image of $T \otimes_S \Omega_{S/R} \to \Omega_{T/R}$, that's exactly the kernel of $\Omega_{T/R} \to \Omega_{T/S}$. □

Remark. Yes, one can ask about the homology of such a functor, however we will not go into this.

In the case that the map $S \to T$ is a surjection, then we would have that $\Omega_{T/S} = 0$ since the S-linear map $d : T \to \Omega_{T/S}$ would be 0, since $dc = 0$ for any c in the image of S. In this situation, we actually get a different but also useful exact sequence.

Proposition 2 (Conormal Sequence). *If $\pi : S \to T$ is a surjection of R-algebras with kernel I, then there is an exact sequence of T-modules*

$$I/I^2 \xrightarrow{\ d\ } T \otimes_S \Omega_{S/R} \xrightarrow{D\pi} \Omega_{T/R} \longrightarrow 0\ ,$$

where the map $D\pi$ maps $a \otimes df$ to adf and d takes $\overline{f}$ to $1 \otimes df$. The module I/I^2 is called the ***conormal module*** *of T/S.*

Proof. First consider the universal derivation $d : S \to \Omega_{S/R}$. Restrict d to I and look at $d(bc)$ for $b \in S$, $c \in I$. We get

$$d(bc) = cdb + bdc,$$

which tells us that d induces an S-linear map $I \to \Omega_{S/R}/I\Omega_{S/R} = (S/I) \otimes \Omega_{S/R} = T \otimes \Omega_{S/R}$. Further, taking $b \in I$ as well, we see that $d(bc)$ goes to 0 in $T \otimes \Omega_{S/R}$, so indeed we have a map

$$I/I^2 \to T \otimes \Omega_{S/R}$$

as desired.

Next, we want to see that the cokernel is given by $D\pi$. Consider the generators and relations that define $T \otimes_S \Omega_{S/R}$. As a T-module, this is generated by df for $f \in S$ subject to the relations of R-linearity and Leibniz rule. This is the same as the generators and relations of $\Omega_{T/R}$ except that df for $f \in I$ are 0 in this new module, which means that the kernel is exactly the image of I/I^2 as we want. □

3 Computation of differentials

Computing differentials is something we actually all already know how to do. For example, we know how to find the differential form of a smooth function on a manifold, and this is just the same idea. For our set up, let us consider S to be a finitely generated R-algebra, say $S = R[x_1, \dots, x_r]/I$, and if $I = (f_1, \dots, f_s)$, then $S \otimes_R \Omega_{R[x_1,\dots,x_r]/R} = S \otimes (\bigoplus R[x_1, \dots, x_r]dx_i) = \bigoplus Sdx_i$ is a free S-module on the generators dx_i. By the conormal sequence,

$$\Omega_{S/R} = \operatorname{coker}(I/I^2 \to \bigoplus Sdx_i)$$

Now, since I/I^2 is a finitely generated S-module (generated by the $\overline{f}_i$'s), we have a surjection $\bigoplus Se_i \to I/I^2$, $e_i \mapsto \overline{f}_i$. Composing with the map $d : I/I^2 \to \bigoplus Sdx_i$, we have a map

$$J : \bigoplus Se_i \to \bigoplus Sdx_i$$

which maps e_i to $\sum_j \frac{\partial f_i}{\partial x_j} dx_j$, i.e. J is represented by the Jacobian matrix $(\partial f_i/\partial x_j)_{ij}$. What this means then is that $\Omega_{S/R}$ is just the cokernel of the Jacobian matrix!

Example 2. If $S = R[x]/(f(x))$, then we have that

$$\Omega_{S/R} = Sdx/(df) = Sdx/(S \cdot f'(x)dx) \simeq S/(f'(x)).$$

Example 3. Let $S = R[x, y, t]/(y^2 - x^2(t^2 - x))$. In this case, the Jacobian matrix is

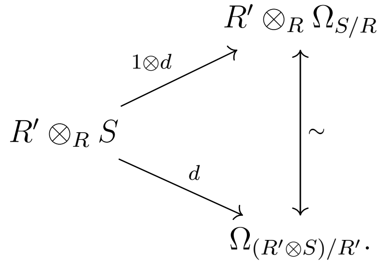
$$J = \begin{pmatrix} -2x(t^2 - x) + x^2 \\ 2y \\ -2x^2 t \end{pmatrix} = \begin{pmatrix} 3x^2 - 2xt \\ 2y \\ -2x^2 t \end{pmatrix}.$$

From here, we see that $\Omega_{S/R}$ is the free S-module generated by dx, dy, and dt modulo the relation

$$(3x^2 - 2xt)dx + 2ydy - 2x^2 tdt = 0.$$

4 How module of differentials interact with other operations

Proposition 3 (Base change)**.** *For any R-algebra R' and S, there is a commutative diagram*



Proof. We will use universal properties to get the vertical maps that we want.

First, note that the map $1 \otimes d : R' \otimes_R S \to R' \otimes_R \Omega_{S/R}$ is a R'-linear derivation, so by the universal property of differentials, we have that there is a map $\Omega_{(R' \otimes_R S)/R} \to \Omega_{S/R}$, which sends $d(r' \otimes s)$ to $r' \otimes ds$.

For the other direction, note that the composite map

$$S = R \otimes_R S \longrightarrow R' \otimes_R S \xrightarrow{d} \Omega_{(R' \otimes S)/R'}$$

is an R-linear derivation, which means that using the universal property again, we get a universal map

$$\Omega_{S/R} \to \Omega_{(R' \otimes S)/R'}$$

sending ds to $d(1 \otimes s)$. Now, since $\Omega_{(R' \otimes S)/R'}$ is further an $R' \otimes S$ module, we can tensor with R' to get an $R' \otimes S$-linear map

$$R' \otimes_R \Omega_{S/R} \to \Omega_{(R' \otimes S)/R'}$$

sending $r' \otimes ds$ to $d(r' \otimes s)$. This is the inverse of the previous map. □

Proposition 4 (Tensor products)**.** *IF $T = \bigotimes_R S_i$ is the tensor product of some R-algebras S_i, then*

$$\begin{aligned} \Omega_{T/R} &\simeq \bigoplus_i (T \otimes_{S_i} \Omega_{S_i/R}) \\ &= \bigoplus_i \left(\left(\bigotimes_{R, i \neq j} S_j \right) \otimes_R \Omega_{S_i/R} \right) \end{aligned}$$

by an isomorphism α satisfying

$$\alpha : d(\ldots \otimes 1 \otimes b_i \otimes 1 \otimes \ldots) \mapsto (\ldots, 0, 1 \otimes db_i, 0, \ldots)$$

where $b_i \in S_i$ occurs in the i-th place of each expression.

Proof. First note that

$$\begin{aligned} T \otimes_{S_i} \Omega_{S_i/R} &= \left(\bigotimes_R S_j \right) \otimes_{S_i} \Omega_{S_i/R} \\ &= \left(\bigotimes_{R, i \neq j} S_j \right) \otimes_R S_i \otimes_{S_i} \Omega_{S_i/R} \\ &= \left(\bigotimes_{R, i \neq j} S_j \right) \otimes_R \Omega_{S_i/R}. \end{aligned}$$

Now, let

$$\Omega = \bigoplus_i \left(\left(\bigotimes_{j \neq i} S_j \right) \otimes_R \Omega_{S_i/R} \right).$$

Write $d_i : S_i \to \Omega_{S_i/R}$ to be the universal derivation on S_i. Now, any element $t \in T$ may be written as a finite sum of terms $\otimes b_i$, where only finitely many of the $b_i \neq 1$. So we have that only finitely many of the maps

$$1 \otimes d_i : T = \left(\bigotimes_{i \neq j} S_j \right) \otimes S_i$$

are nonzero on t, so we may define a map $e : T \to \Omega$ to be the sum

$$\sum_i 1 \otimes d_i.$$

Since each of the $1 \otimes d_i$ are derivations, the map e must then also be a derivation. So we have an induced T-module homomorphism

$$\alpha : \Omega_{T/R} \to \Omega$$

mapping $d(\bigotimes_i b_i)$ to $e(\bigotimes_i b_i)$. We want to see then that this α is an isomorphism.

To see that α is an isomorphism, we need to produce an inverse map. To produce an inverse, note that for each S_i, the composite map $S_i \to T \to \Omega_{T/R}$ is an R-linear derivation $S_i \to \Omega_{T/R}$, and thus by the universal property, we have an S_i-linear map $\Omega_{S_i/R} \to \Omega_{T/R} : db_i \mapsto d(1 \otimes b_i)$, where 1 is the identity in $\bigotimes_{i \neq j} S_j$. Since $\Omega_{T/R}$ is a T-module, this extends to a T-linear map by tensoring, and so we get a map

$$\beta_i : T \otimes_{S_i} \Omega_{S_i/R} \to \Omega_{T/R}$$

with $\beta_i(1 \otimes d_i b_i) = d(1 \otimes b_i)$. Summing the β_i's together, we get a map $\Omega = \bigoplus_i T \otimes_{S_i} \Omega_{S_i/R} \to \Omega_{T/R}$ that is the inverse of α as desired. ☐

Geometrically, this can be interpreted in the following way. We have two 'spaces' X_1, and X_2, then locally, the product $X_1 \times X_2$ should look like spectrum of the tensor product of two rings. We have that looking at the fiber of the (co)tangent bundle at the point $(p_1, p_2) \in X_1 \times X_2$, we should have that $T_{(p_1,p_2)} X_1 \times X_2$ be canonically isomorphic to $T_{p_1} X_1 \oplus T_{p_2} X_2$.

Proposition 5. *If $T = S[x_1, \ldots, x_r]$ is a polynomial ring over an R-algebra S, then*

$$\Omega_{T/R} \simeq (T \otimes_S \Omega_{S/R}) \oplus \left(\bigoplus_i T dx_i \right).$$

Proof. Let $T' = R[x_1, \ldots, x_r]$. Write $T = S \otimes_R T'$, we can apply the above to get that

$$\Omega_{T/R} \simeq (T \otimes_S \Omega_{S/R}) \oplus (T \otimes_{T'} \Omega_{T'/R}).$$

Now, we have that $\Omega_{T'/R} \simeq \bigoplus_i S dx_i$, so we have that $T \otimes_{T'} \bigoplus_i S dx_i = \bigoplus_i T dx_i$. This gives us what we want. ☐

Our next goal is to see that module of differentials commute with localization and direct limits (i.e. colimits). To do that, we first need the notion of coequalizers. Here we only need the notion for R-algebras.

Definition 3. Given a pair of R-algebra morphisms $\psi, \psi' : S_1 \to S_2$, the **coequalizer** of ψ and ψ' is the algebra $T = S_2/I$, where I is the ideal generated by the relations $\psi(a) - \psi'(a) = 0$ for all $a \in S_1$.

Lemma 1. *If T is the coequalizer of a pair of maps $\psi, \psi' : S_1 \to S_2$ then there is a right exact sequence of T-modules*

$$T \otimes_{S_1} \Omega_{S_1/R} \xrightarrow{T\otimes D\psi - T\otimes D\psi'} T \otimes_{S_2} \Omega_{S_2/R} \longrightarrow \Omega_{T/R} \longrightarrow 0.$$

Proof. By the conormal sequence, we have that $\Omega_{T/R}$ is $T \otimes_{S_2} \Omega_{S_2/R}$ modulo the submodule generated by the elements $d(\psi(a) - \psi'(a))$ for $a \in S_1$, which is exactly the image of the map $T \otimes D\psi - T \otimes D\psi'$. □

To see that module of differentials commute with colimits, we will need a basic categorical fact.

Theorem 1. *If coproducts of sets of objects and coequalizers of pairs of morphisms exist in the category $\mathcal{A}$, then all all colimits of functors from small categories exist in $\mathcal{A}$. Further, any functor on $\mathcal{A}$ that preserves coproducts and coequalizers preserves all colimits over small categories.*

Theorem 2 (Differentials commute with colimits)**.** *Let $\mathcal{B}$ be a diagram in the category of R-algebras. Set $\varinjlim \mathcal{B} = T$ (just say that $\mathcal{B}$ is a direct system in the category of R-algebras). If F is the functor from $\mathcal{B}$ to the category of T-modules taking an object S to $T \otimes_S \Omega_{S/R}$ and a morphism $\varphi : S' \to S$ to morphism $1 \otimes D\varphi : T \otimes_S (S \otimes_{S'} \Omega_{S'/R}) \to T \otimes_S \Omega_{S/R}$, then*

$$\Omega_{T/R} = \varinjlim F.$$

Proof. It is a fact that colimits are a combination of coequalizers and coproducts (i.e. tensor products). The details of this construction is a bit too involved for the scope of this report. We have already proven that the construction of module of differentials preserve all coproducts and coequalizers, so the result follows from these. □

From here, we can see that differentials commute with localization.

Proposition 6 (Module of differentials commute with localization)**.** *If S is an R-algebra and U is a multiplicatively closed subset of S, then*

$$\Omega_{S[U^{-1}]/R} \simeq S[U^{-1}] \otimes_S \Omega_{S/R}$$

in such a way that $d(1/s) = (-1/s^2)ds$ for $s \in U$.

Proof. We will start with the case that $U = \{1, s, s^2, \ldots\}$, i.e. powers of s. So we have that $S[U^{-1}] = S[x]/(sx - 1)$. We see that

$$\Omega_{S[U^{-1}]/R} = (S[U^{-1}]\Omega_{S/R} \oplus S[U^{-1}]dx)/(S[U^{-1}]d(sx-1))$$

by combining what happens with $S[x]$ with what happens with quotients. Now, since $d(sx - 1) = xds + sdx$ and since s is a unit in $S[U^{-1}]$, we see that

$$\Omega_{S[U^{-1}]/R} = (S[U^{-1}]\Omega_{S/R} \oplus S[U^{-1}]dx)/(S[U^{-1}]dx) = S[U^{-1}]\Omega_{S/R},$$

where dx is identified with $-(x/s)ds$. Now if we think of x as s^{-1}, then this reads as $(-1/s^2)ds$, as desired.

The general case follows by a direct limit argument. If $\mathcal{B}$ is the diagram of R-algebras whose objects are localisations $S[s^{-1}]$ for $s \in U$ with maps $S[s^{-1}] \to S[(st)^{-1}]$ given by the natural localization maps for $s, t \in U$, then $S[U^{-1}] = \varinjlim \mathcal{B}$, and so using the fact that colimits commute with differentials, we have that

$$\Omega_{S[U^{-1}]/R} = \varinjlim_{s \in U} S[U^{-1}] \otimes_{S[s^{-1}]} \Omega_{S[U^{-1}]/R} = \Omega_{S/R}[U^{-1}] = S[U^{-1}] \otimes \Omega_{S/R}.$$

□

This result is very nice because it tells us that the sheaf associated to modules of differentials behave exactly as you would expect them to behave.

Proposition 7 (Differentials commute with direct products)**.** *If $S_1, \ldots, S_n$ are R-algebras and $S = \prod_i S_i$, then*

$$\Omega_{S/R} = \prod_i \Omega_{S_i/R}.$$

Proof. If e_i is the idempotent of S that is the unit of S_i and D is a derivation of S to S-module M, then $De_i = 0$ and so

$$D(e_i f) = e_i Df.$$

Thus we have that D maps $S_i = e_i S$ to $M_i = e_i M$ and corresponds to a unique map $\Omega_{S_i/R} \to M_i$. It follows then that $\prod_i \Omega_{S_i/R}$ has the universal property that characterizes $\Omega_{S/R}$. □

5 Big theorem about modules of differentials

Really the big idea behind this is that we want to detect smoothness of algebraic varieties of schemes in general. The notion of smoothness of varieties is encapsulated by the notion of a regular local ring.

Definition 4. A local ring $(R, \mathfrak{m})$ is **regular** if $\mathfrak{m}$ can be generated by $\dim R$ elements.

This notion corresponds to smoothness because it says that basically the tangent space at $\mathfrak{m}$ has the same dimension as the variety.

Definition 5. An (abstract) variety X over an algebraically closed field k is **nonsingular** if all its local rings are regular local rings.

The following two theorems describe the relationship between regularity and module of differentials. We will skip the proofs since they involve more machinery than what can be done here.

Theorem 3 (Jacobian Criterion)**.** *Let $S = k[x_1, \dots, x_r]$ and $I = (f_1, \dots, f_s)$ and $R = S/I$. Let P be a prime ideal of S containing I and write $\kappa(P) = K(S/P)$. Let c be the codimension of I_P in S_P.*

(a) The Jacobian matrix

$$J = (\partial f_i / \partial x_j)_{ij}$$

taken modulo P has rank $\leq c$.

(b) If $\operatorname{char} k = p > 0$, assume that $\kappa(P)$ is separable over k. R_P is a regular local ring if and only if the matrix J, taken modulo P, has rank exactly c.

Theorem 4. *Let S, I, R and c be as before. Assume further now that P is a prime ideal of S containing I such that $\kappa(P) = K(S/P)$ is now separable over k. R_P is a regular local ring if and only if the module $\Omega_{R/k}$ is locally free at P of rank $r - c$.*

References

[1] David Eisenbud. *Commutative Algebra With a View Toward Algebraic Geometry*, volume 150. Springer Science & Business Media, 2013.

[2] Robin Hartshorne. *Algebraic Geometry*, volume 52. Springer Science & Business Media, 2013.